Multilateration

Previous derivations on this topic were made in years passed to enable calculations with limited numbers of hydrophone channels. This page provides a generalized algorithm for locating a pinger in 3D in a robust, linear manner with an arbitrary number of arbitrarily placed hydrophones. This solution is an unambiguous, precise result that evaluates at all distances and does not require that the signal originate in the far-field.

This method of calculation requires 5 or more hydrophones. Substituting a depth sensor for one of the required hydrophones is possible, and covered below. The submarine does *not* need to be held level to make use of the depth sensor.

The exact math to implement for the solution can be found by scrolling directly to the bottom of this page.

Problem Setup

Apart from a minimum number of hydrophones, there are no constraints on the set-up of the system. One hydrophone h_0 is used as a reference and lies at the origin by definition. A second hydrophone h_1 serves as a static reference point against the origin hydrophone. It may still be placed at an arbitrary location. All other hydrophones h_n for $n>1$ are evaluated against these two hydrophones.

h_0 is at location $h_0 = (0,0,0)$
h_1 is at location $h_1 = (x_1,y_1,z_1)$
h_n is at location $h_n = (x_n,y_n,z_n)$

The ping source is at location $$ p_{pinger} = (X,Y,Z) $$

When a ping is received by the hydrophones, the hardware outputs differences in time-of-arrival from the origin, Δt_n, which corresponds to the difference in time between when the ping was received by h_0 and h_n, which is a function of their pathlengths to the pinger.

$$ \Delta t_n = \frac{\sqrt{(X-x_n)^2 + (Y-y_n)^2 + (Z-z_n)^2} - \sqrt{X^2 + Y^2 + Z^2}}{c_s} $$

These time differences Δt_n are multiplied by the speed of sound c_s (~1500m/s) to find path-length differences Δr_n

$$ \Delta r_n = \Delta t_n * c_s = \sqrt{(X-x_n)^2 + (Y-y_n)^2 + (Z-z_n)^2} - \sqrt{X^2 + Y^2 + Z^2} $$

With these measured values, $x_n,y_n,z_n,\Delta r_n,$ for each hydrophone, we can fill in all values for the linear expression between our measurements, and theposition of the pinger (X,Y,Z).

Linear Equation

The linear equation takes the form:

$$ A_nx = C_n - B_n \\ \begin{bmatrix} \begin{vmatrix} 2x_1 & 2x_n \\ \Delta r_1 & \Delta r_n \\ \end{vmatrix}, & \begin{vmatrix} 2y_1 & 2y_n \\ \Delta r_1 & \Delta r_n \\ \end{vmatrix}, & \begin{vmatrix} 2z_1 & 2z_n \\ \Delta r_1 & \Delta r_n \\ \end{vmatrix} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} (x_1^2+y_1^2+z_1^2) & (x_n^2+y_n^2+z_n^2) \\ \Delta r_1 & \Delta r_n \\ \end{vmatrix} \end{bmatrix} - \begin{bmatrix} \begin{vmatrix} \Delta r_1^2 & \Delta r_n^2 \\ \Delta r_1 & \Delta r_n \\ \end{vmatrix} \end{bmatrix} $$

This linear relationship is valid for all $n>1$. Note that the straight lines in the equations denote 2×2 determinants.

In order to get a definitive answer for the three unknown values $X, Y,$ and $Z,$ we need at least three of these linear relations. Hence we need at least 3 hydrophones apart from the origin hydrophone h_0 and the static reference hydrophone h_1. Or a total of 5 hydrophones minimum.

$$ Ax = C - B \\ \begin{bmatrix} A_{x2} & A_{y2} & A_{z2} \\ A_{x3} & A_{y3} & A_{z3}\\ A_{x4} & A_{y4} & A_{z4}\\ \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ \end{bmatrix} = \begin{bmatrix} C_2 \\ C_3 \\ C_4\\ \end{bmatrix} - \begin{bmatrix} B_2 \\ B_3 \\ B_4\\ \end{bmatrix} \\ \begin{bmatrix} \begin{vmatrix} 2x_1 & 2x_2 \\ \Delta r_1 & \Delta r_2 \\ \end{vmatrix}, & \begin{vmatrix} 2y_1 & 2y_2 \\ \Delta r_1 & \Delta r_2 \\ \end{vmatrix}, & \begin{vmatrix} 2z_1 & 2z_2 \\ \Delta r_1 & \Delta r_2 \\ \end{vmatrix} \\ \begin{vmatrix} 2x_1 & 2x_3 \\ \Delta r_1 & \Delta r_3 \\ \end{vmatrix}, & \begin{vmatrix} 2y_1 & 2y_3 \\ \Delta r_1 & \Delta r_3 \\ \end{vmatrix}, & \begin{vmatrix} 2z_1 & 2z_3 \\ \Delta r_1 & \Delta r_3 \\ \end{vmatrix} \\ \begin{vmatrix} 2x_1 & 2x_4 \\ \Delta r_1 & \Delta r_4 \\ \end{vmatrix}, & \begin{vmatrix} 2y_1 & 2y_4 \\ \Delta r_1 & \Delta r_4 \\ \end{vmatrix}, & \begin{vmatrix} 2z_1 & 2z_4 \\ \Delta r_1 & \Delta r_4 \\ \end{vmatrix} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} (x_1^2+y_1^2+z_1^2) & (x_2^2+y_2^2+z_2^2) \\ \Delta r_1 & \Delta r_2 \\ \end{vmatrix} \\ \begin{vmatrix} (x_1^2+y_1^2+z_1^2) & (x_3^2+y_3^2+z_3^2) \\ \Delta r_1 & \Delta r_3 \\ \end{vmatrix} \\ \begin{vmatrix} (x_1^2+y_1^2+z_1^2) & (x_4^2+y_4^2+z_4^2) \\ \Delta r_1 & \Delta r_4 \\ \end{vmatrix} \\ \end{bmatrix} - \begin{bmatrix} \begin{vmatrix} \Delta r_1^2 & \Delta r_2^2 \\ \Delta r_1 & \Delta r_2 \\ \end{vmatrix} \\ \begin{vmatrix} \Delta r_1^2 & \Delta r_3^2 \\ \Delta r_1 & \Delta r_3 \\ \end{vmatrix} \\ \begin{vmatrix} \Delta r_1^2 & \Delta r_4^2 \\ \Delta r_1 & \Delta r_4 \\ \end{vmatrix} \\ \end{bmatrix} $$

Which would then be resolved as $$ X = A^{-1}(C-B) \\ \begin{bmatrix} X \\ Y \\ Z \\ \end{bmatrix} = \begin{bmatrix} A_{x2} & A_{y2} & A_{z2} \\ A_{x3} & A_{y3} & A_{z3}\\ A_{x4} & A_{y4} & A_{z4}\\ \end{bmatrix}^{-1} \left(\begin{bmatrix} C_2 \\ C_3 \\ C_4\\ \end{bmatrix} - \begin{bmatrix} B_2 \\ B_3 \\ B_4\\ \end{bmatrix} \right) $$

Rotation

In the event the submarine is rotated off of the default orientation $R(\psi,\phi,\theta) = R(0,0,0)$ then the relative estimation of the pinger's position will be off, and a correction must be made to get the absolute coordinates $p_{pinger}=(X,Y,Z)$.

If the array is rotated by R then the relative position of the pinger has been rotated in the local frame by the inverse R^T

$$ R^T * X_{global} = X_{local} = A^{-1}(C-B) $$

Therefore:

$$ X_{global} = RA^{-1}(C-B) $$

Which is our desired solution. Working backwards, we can also properly write the linear system for each individual hydrophone as: $$ A_nR^TX_{global} = (C_n-B_n) $$

Depth Sensor Substitution

Now that the evaluation of (X,Y,Z) is part of a linear system, it is easy to see how direct measurements of any of the states can substitute for hydrophone measurements.

$$ \begin{bmatrix} A_{x2} & A_{y2} & A_{z2} \\ A_{x3} & A_{y3} & A_{z3}\\ 0 & 0 & 1\\ \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ \end{bmatrix} = \begin{bmatrix} C_2 \\ C_3 \\ z_{pinger}\\ \end{bmatrix} - \begin{bmatrix} B_2 \\ B_3 \\ z_{sub}\\ \end{bmatrix} \\ $$

Which requires only 4 hydrophones, $h_0,h_1,h_2,$ and h_3 plus the depth sensor measuring z_{sub} to compared against the known depth of the pinger z_{pinger}.

However, the depth sensor will always measure the absolute depth of the submarine regardless of the submarine's orientation $R(\psi,\phi,\theta)$. Thus the difference measured will always be $z_{pinger}-z_{sub} = Z_{global}$.

Thus in order to properly set up the system, we must use the formula derived in the Rotation section:

$$ A_nR^TX_{global} = (C_n-B_n) $$

Which properly applied gives us the full equation:

$$ \begin{bmatrix} \begin{bmatrix} A_{x2} & A_{y2} & A_{z2} \\ A_{x3} & A_{y3} & A_{z3}\\ \end{bmatrix} * \begin{bmatrix} R \end{bmatrix}^T \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ \end{bmatrix} = \begin{bmatrix} C_2 \\ C_3 \\ z_{pinger}\\ \end{bmatrix} - \begin{bmatrix} B_2 \\ B_3 \\ z_{sub}\\ \end{bmatrix} \\ $$

Which finally lets us calculate the full solution:

Full Solution

$$ \begin{bmatrix} X \\ Y \\ Z \\ \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{x2} & A_{y2} & A_{z2} \\ A_{x3} & A_{y3} & A_{z3}\\ \end{bmatrix} * \begin{bmatrix} R \end{bmatrix}^T \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{bmatrix}^{-1} \left(\begin{bmatrix} C_2 \\ C_3 \\ z_{pinger}\\ \end{bmatrix} - \begin{bmatrix} B_2 \\ B_3 \\ z_{sub}\\ \end{bmatrix} \right) \\ $$

With polar bearings:

$$ Azimuth = arctan\left(\frac{Y}{X}\right), \qquad Inclination = arctan\left(\frac{Z}{\sqrt{X^2+Y^2}}\right), \qquad Range = \sqrt{X^2 + Y^2 + Z^2} \\ $$

given: $$ R = R(\psi,\phi,\theta), \qquad \Delta r_n = \Delta t_n * c_s, \qquad h_n = (x_n,y_n,z_n)\\ A_{xn} = det\begin{vmatrix} 2x_1 & 2x_n \\ \Delta r_1 & \Delta r_n \end{vmatrix}, \qquad A_{yn} = det\begin{vmatrix} 2y_1 & 2y_n \\ \Delta r_1 & \Delta r_n \end{vmatrix}, \qquad A_{zn} = det\begin{vmatrix} 2z_1 & 2z_n \\ \Delta r_1 & \Delta r_n \end{vmatrix}, \\ B_n = det\begin{vmatrix} \Delta r_1^2 & \Delta r_n^2 \\ \Delta r_1 & \Delta r_n \end{vmatrix}, \qquad C_n = det\begin{vmatrix} (x_1^2+y_1^2+z_1^2) & (x_n^2+y_n^2+z_n^2) \\ \Delta r_1 & \Delta r_n \end{vmatrix} $$